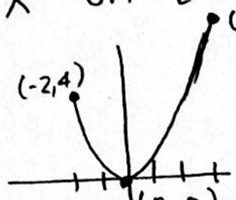
Lesson 20: Absolute Extrema

Def The absolute maximum (absolute minimum) of $f(x)$ is the largest (smallest) value of $f(x)$ (y-value) on a given interval. These are called the absolute extrema.

Where can absolute extrema occur?

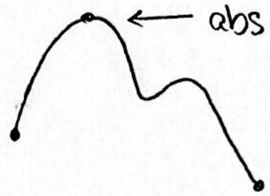
Ex 1 $y=x^2$ on $[-2,3]$



(3,9) ← absolute max (endpoint of $[-2,3]$)

(0,0) ← absolute min (also a relative min)

Ex 2



← abs. max (rel max)

← abs. min (endpoint)

Fact If $f(x)$ is continuous on a closed interval $[a,b]$, then it has an abs. max. and an abs. min. These occur at ~~relative extrema~~ CV's or endpoints.

true!

To find the absolute extrema of $f(x)$ on $[a,b]$:

1. Find CV's of $f(x)$ in $[a,b]$
2. Evaluate $f(x)$ at each of the CV's (from 1) and endpoints.
3. Determine the extrema.

Ex 3 $f(x) = x^3 - 3x$ on $[0,3]$

$$\left.\begin{aligned} f'(x) &= 3x^2 - 3 \overset{set}{=} 0 \\ 3(x^2-1) &= 0 \\ 3(x+1)(x-1) &= 0 \end{aligned}\right\} \text{warmup!}$$

CV's: $\cancel{x=-1}$, $x=1$

x	0	1	3
$f(x)$	0	-2	18
		↑ abs min	↑ abs max

Ex 4 $f(x) = xe^{-x}$ on $[-1,2]$

$$\begin{aligned} f'(x) &= e^{-x} + xe^{-x}(-1) \\ &= e^{-x} - xe^{-x} \overset{set}{=} 0 \\ & e^{-x}(1-x) = 0 \end{aligned}$$

CV's: no CV's (e^{-x}), $x=1$

x	-1	1	2
$f(x)$	$-e$	$\frac{1}{e}$	$\frac{2}{e^2}$
	↑ abs min	$\approx .3678$ abs max	$\approx .2706$

The fact may not be true if we have an open interval (a,b) or a half-open interval $[a,b)$ or $(a,b]$.

But in some cases we still have absolute extrema.

Ex 5 $y = x^2 - 2x$ on $(0,4)$

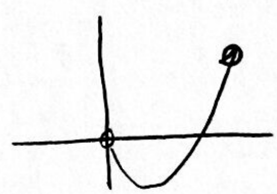
$y' = 2x - 2 \overset{set}{=} 0$

CV: $x = 1$

y': $-$ on $(0,1)$, $+$ on $(1,4)$

x: 0, 1, 4

↑ abs. min at $x=1$



$\boxed{\text{absolute min at } (1,-1)}$

Ex 6 $\quad f(x) = \dfrac{\ln x}{x}$ on $[2,3]$

$$f'(x) = \frac{x(\frac{1}{x}) - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

$1 - \ln x = 0$ $\qquad\qquad$ $x^2 = 0$

$\ln x = 1$ $\qquad\qquad$ $\cancel{x=0}$ not in domain

$x = e$

x	2	e	3
$f(x)$	$\frac{\ln 2}{2}$	$\frac{1}{e}$	$\frac{\ln 3}{3}$
	$\approx .3465$	$\approx .3678$	$\approx .3662$

$\boxed{\text{absolute max at } x = e \quad \text{absolute min at } x = 2}$

Ex 7 $\quad f(x) = \dfrac{x}{x^2+1}$ on $(0,4]$ ← half-open, so not guaranteed a max or min!

$$f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$$= \frac{(1-x)(1+x)}{(x^2+1)^2}$$

CV's: $x = 1, \cancel{-1}$

f'		$+$		$-$	
x	0		1		4

$\boxed{\text{So there is an absolute max at } x=1}$